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Non-linear micropolar and classical continua for anisotropic discontinuous materials

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Abstract

Non-linear Cosserat and Cauchy anisotropic continua equivalent to masonry-like materials, like brick/block masonry, jointed rocks, granular materials or matrix/particle composites, are presented.

An integral procedure of equivalence in terms of mechanical power has been adopted to identify the effective elastic moduli of the two continuous models starting from a Lagrangian system of interacting rigid elements. Non-linear constitutive functions for the interactions in the Lagrangian system are defined in order to take into account both the low capability to carry tension and the friction at the interfaces between elements. The non-linear problem is solved through a finite element procedure based on the iterative adjustment of the continuum constitutive tensor due to the occurrence of some limit situation involving the contact actions of the discrete model.

Differences between the classical and the micropolar model are investigated with the aid of numerical analyses carried out on masonry walls made of blocks of different size. The capability of the micropolar continuum to discern, unlike the classical continuum, the behaviour of systems made of elements of different size is pointed out. It is also shown that for anisotropic materials, even in the elastic case, the micropolar solution in general does not tend to the classical solution when the size of the elements vanishes.

Keywords: Block materials; Equivalent continua; Anisotropic Cosserat continua

1. Introduction

In this work, constitutive micropolar and classical models for discontinuous and heterogeneous materials are proposed. The materials considered, with an internal structure at the mesoscopic level, can be described as systems of rigid elements interacting through a medium incapable to carry tension and resistant to sliding by friction. Brick/block masonry, rock assemblages, matrix/particle materials are examples of such materials.

The behaviour of these media depends on the mechanical properties of their constituents and is strongly influenced by the shape, the size, the orientation and the arrangement of the units. A detailed modelling,

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based on the description of these materials as Lagrangian systems, is the most obvious way of taking into account the above features (Baggio and Trovalusci, 1993; Baggio and Trovalusci, 2000; Camborde et al., 2000). However, such an approach does not always apply to systems characterised by many degrees of freedom and it is often advisable to choose a macroscopic modelling in terms of continuum.

The continuous modelling is effective in reproducing the macroscopic behaviour of a medium, which is actually heterogeneous and discontinuous, when it enables to consider the mechanical properties of the components, as well as the geometry and the texture of the elements. Recently (Rots et al., 1998; Kyoya et al., 1999; Banks-Sills et al., 1997), considerable effort has been made towards estimating the average elastic properties and the failure criteria of equivalent continua within the context of the homogenisation theories. These models have, for the most part, constitutive relations of a classical continuum that do not include any length parameter of the microstructure. In addition, according to the standard techniques of homogenisation, the boundary problem must be solved on a representative periodical cell with vanishing dimensions. Whenever the characteristic length of the macrostructure is comparable to the characteristic length of the microstructure, the classical continuum description based on the microscopic solution on a dimensionless domain reaches its limit.

The standard Cauchy continuum cannot discern the behaviour of structures composed of similar elements with different size and, consequently, it cannot account for possible scale effects. A classical related issue lies in problems of localisation of strain-softening materials (Read and Hegemier, 1984), where the lack of an internal length parameter in the constitutive functions produces a loss of hyperbolicity in the equations of motion. As a result, the numerical solutions strongly depend on the adopted discretisation. Several are the solutions proposed to this problem: from the use of viscous models (Sluys and Wang, 1998) to the employment of grade two materials and non-local models (Peerlings et al., 1996; Svedberg and Runesson, 1998; Polizzotto et al., 1998). They are all related to the introduction of specific constitutive requirements involving spatial or temporal derivatives other than the first order in the equations of motion. These additional terms, related to an internal length parameter typical of the internal structure of the medium, define a "non-simple" Cauchy continuum. Through them, the problem can be regularised and becomes well posed. Another debated issue concerns the thermodynamic compatibility, which is clearly violated when the additional terms have no counterpart, in terms of work expended, in the governing equations (Gurtin, 1965). Nevertheless, also when the material has no softening behaviour or is elastic, if the gradient of strain and the mutual rotations of the elements become significant—in case of concentrated loads, openings and other geometrical discontinuities—different behaviour among structures made of small or large elements must be appreciated.

Internal length parameters can be included in the constitutive relations of a continuous model, for example by defining field descriptors in addition to the classical displacement field descriptor, and consequently by determining additional measures of strain and stress. In this way, models called multi-field continua (Mariano and Trovalusci, 1999) will be obtained. These continua, derived by Capriz (1989), are continua with any kind of local microstructure and represent an extension of the continua with kinematical microstructure defined for example in the classical work of Mindlin (1964). It can be easily shown that such models are thermodynamically compatible. [1] Besides, it can be shown that their formulation allows to regularise possible ill-conditionings in the differential equations of motions. [2]

[1] In particular Capriz (1985) demonstrated that for materials with microstructure and for second gradient materials—the latter are considered special cases of the former with specific internal constraints—the presence of additional terms in the balance of energy ensures the thermodynamic compatibility.

[2] This possibility is related to the presence of additional field descriptors and of their gradients in the balance equations, for example the microtation and its gradient in the micropolar model, which is a continuum with local rigid microstructure. For the isotropic Cosserat continuum this possibility has been widely investigated in case of shear bands (Mühlhaus and Vardoulakis, 1987; Willam et al., 1995; Elers and Volk, 1997; Iordache and Willam, 1998).

In earlier works, the authors (Masiani and Trovalusci, 1995; Masiani and Trovalusci, 1996; Trovalusci and Augusti, 1998; Mariano and Trovalusci, 1999; Trovalusci and Masiani, 1999) have shown, within the elastic frame, that information related to the actual heterogeneous and discontinuous nature of a material can be preserved using a macroscopic description in terms of continuous model with microstructure. This model corresponds, in terms of mechanical power, to a proper Lagrangian model of the material. In particular, it has been acknowledged that a multi-field continuum model for masonry-like materials should have a polar microstructure. The analyses on walls made of elements with various geometry and textures showed that the Cosserat model allows discerning the behaviour of systems made of elements of different shape, size and arrangement. Moreover, since the strain and the stress tensors are not symmetric, this model can describe the asymmetries in the shear behaviour along different planes. This property proved useful for orthotropic materials like common masonry with regularly spaced bricks.

This work explores the differences between the micropolar and the classical solution within the linear elastic and the non-linear constitutive frames. The non-linear behaviour of masonry materials is considered in order to take into account the low capability to carry tension and the friction at the interfaces between elements. Non-linear, and non-softening, constitutive relations for the equivalent Cosserat continuum are then obtained by defining suitable response functions for the interactions between elements in the Lagrangian system. Moreover, non-linear constitutive relations are derived for the classical continuum equivalent to the same discrete assembly. Differences between the two solutions, except particular cases of deformations like the uniform extension, are often remarkable and essentially related not only to the size of the elements but also to their shape and disposition, that is to the symmetry class of the material.

The non-linear problem, for the micropolar and the classical material, is solved through a finite element procedure in a two-dimensional frame. A plane stress element corrected with additional degrees of freedom (the in-plane rotations) is used for the Cosserat model. The algorithm performed is based on the iterative adjustment of the micropolar or the classical constitutive tensors. The possibility to check the material at microstructural level allows working in the space of the contact actions. The definition of yield surfaces for the equivalent continuum becomes therefore unnecessary.

The differences between the classical and the micropolar model emerge from various numerical analyses carried out on masonry walls made of bricks of different size. These same differences increase in the non-linear case. The Cosserat continuum, namely, performs better, even in the linear elastic frame, when the mutual rotations among bricks are important: in this case, the solution strongly depends on the internal length size. In general, the micropolar model proves effective with concentrated load problems or with problems related to geometrical discontinuities, non-linear problems involving ruptures with relative rotations and settlements of the units and with problems related with shear behaviour.

2. Cosserat and Cauchy linear elastic materials

The mechanical behaviour of heterogeneous materials with respect to the observation scale is often related to the particular kind of internal microstructure. When the macroscopic length parameter—a load print, the size of an opening or other characteristic dimension of the body—is comparable to the size of the microheterogeneity, the response of the material is strongly affected by the geometry and arrangement of the units. Adopting a model with local microstructure we can take into account the actual discontinuous and heterogeneous nature of a medium without losing the benefits of the continuum modelling.

As discussed in the above mentioned works of Masiani and Trovalusci (1996) and Trovalusci and Masiani (1999), the simplest kind of microstructure which can be used to describe masonry-like materials is the rigid microstructure of the Cosserat type. To this purpose, the constitutive relations of a micropolar linear elastic medium were obtained through an integral equivalence procedure starting from the description, in a linearised context, of a Lagrangian system of rigid bodies interacting through linear elastic interfaces.

This system consists of rigid elements interacting two by two, at the contact point C, through contact forces, represented by the vector $\mathbf{t}_C$, and contact couples, represented by the skew-symmetric tensor $\mathbf{T}_C$. The strain measures of the assembly are the relative displacement at C between each pair of adjacent elements, represented by the vector $\mathbf{u}_C$, and the relative rotation between the two elements, represented by the skew-symmetric tensor $\mathbf{W}_C$,

$$\begin{aligned} \mathbf{u}_C &= \mathbf{u}_B - \mathbf{u}_A + \mathbf{W}_B(C-B) - \mathbf{W}_A(C-A), \\ \mathbf{W}_C &= \mathbf{W}_B - \mathbf{W}_A, \end{aligned} \tag{1}$$

where the vectors $\mathbf{u}_A$ and $\mathbf{u}_B$ represent the displacements of the centres of the elements, A and B, and the skew-symmetric tensors $\mathbf{W}_A$ and $\mathbf{W}_B$ represent the rotations of the elements (Fig. 1).

Considering a small representative periodical neighbourhood of the discrete system (module), the mean work of the contact actions over the module is

$$\pi(\mathbf{u}_C, \mathbf{W}_C) = \frac{1}{V}\sum_C \left(\mathbf{t}_C \cdot \mathbf{u}_C + \frac{1}{2}\mathbf{T}_C \cdot \mathbf{W}_C\right), \tag{2}$$

where V is the volume of the module, while the summation is extended to the contact points among the elements of the module.

In order to identify a continuous equivalent model, the kinematical quantities are assumed as homogeneous in the module

$$\begin{aligned} \mathbf{u}_A &= \mathbf{u}(X) + \mathbf{H}(X)(A-X), \\ \mathbf{W}_A &= \mathbf{W}(X) + \mathbf{R}(X)(A-X), \end{aligned}$$

where $\mathbf{u}(X)$ and $\mathbf{W}(X)$ are respectively the displacement and the rotation of a point X and $\mathbf{H} = \operatorname{grad}\mathbf{u}$ and $\mathbf{R} = \operatorname{grad}\mathbf{W}$. By substituting Eq. (3) into (1) it follows

$$\begin{aligned} \mathbf{u}_C &= \mathbf{U}(B-A) + (\mathbf{R}(B-X))(C-B) - (\mathbf{R}(A-X))(C-A), \\ \mathbf{W}_C &= \mathbf{R}(B-A), \end{aligned} \tag{3}$$

where $\mathbf{U} = \mathbf{H} - \mathbf{W}$. Then the expression of the mean work in terms of the continuum fields $\mathbf{u}$ and $\mathbf{W}$ writes

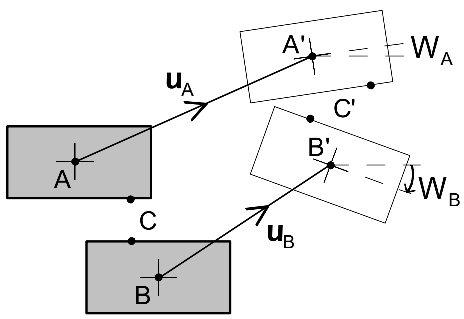

Fig. 1. Kinematical descriptors for the Lagrangian system.

$$\pi(\mathbf{U},\mathbf{R}) = \frac{1}{V}\left\{ \sum_C (\mathbf{t}_C \otimes (B-A)) \cdot \mathbf{U} + \sum_C \left\{ \mathbf{t}_C \otimes [(C-B)\otimes(B-X) - (C-A)\otimes(A-X)] + \frac{1}{2}\mathbf{T}_C \otimes (B-A) \right\} \cdot \mathbf{R} \right\}. \tag{4}$$

If the fields **U** and **R** are identified with the strain measures of a micropolar continuum, represented respectively by the second order strain tensor and the third order microrotation gradient tensor, the work expended per unit volume by the stress measures of the continuum is

$$p(\mathbf{U},\mathbf{R}) = \mathbf{S}\cdot\mathbf{U} + \frac{1}{2}\mathbf{C}\cdot\mathbf{R}, \tag{5}$$

where the second order tensor **S** and the third order tensor **C** represent respectively the Cosserat stress and couple-stress tensor. By asking the equivalence between $\pi(\mathbf{U},\mathbf{R})$ and $p(\mathbf{U},\mathbf{R})$ for any **U** and **R**, the stress measures of the micropolar continuum as functions of the contact actions of the module and of its geometry are obtained:

$$\begin{aligned} \mathbf{S}(X) &= \frac{1}{V}\sum_C \mathbf{t}_C \otimes (B-A), \\ \mathbf{C}(X) &= \frac{1}{V}\sum_C \{2\mathbf{t}_C \otimes [(C-B)\otimes(B-X) - (C-A)\otimes(A-X)] + \mathbf{T}_C \otimes (B-A)\}. \end{aligned} \tag{6}$$

As response functions for the contact actions of the module linear elastic functions are assumed as

$$\begin{aligned} \mathbf{t}_C &= \mathbf{K}_C \mathbf{u}_C, \\ \mathbf{T}_C &= \mathbf{K}^r_C \mathbf{W}_C, \end{aligned} \tag{7}$$

where the components of the second and fourth order symmetric tensors $\mathbf{K}_C$ and $\mathbf{K}^r_C$ are the stiffness of the material interposed among the elements. In particular, $\mathbf{K}_C$ contains the normal, shear parameters as well as the dilatancy parameter, while $\mathbf{K}^r_C$ includes the drilling parameters. Through these relations we obtain the linear elastic constitutive equations of the micropolar equivalent model

$$\begin{aligned} \mathbf{S} &= \mathsf{A}(\mathbf{U}) + \mathsf{B}(\mathbf{R}), \\ \mathbf{C} &= \mathsf{C}(\mathbf{U}) + \mathsf{D}(\mathbf{R}). \end{aligned} \tag{8}$$

The components of the constitutive tensors A, B, C and D, of the fourth, fifth, fifth and sixth order respectively, depend on the elastic constants of the matrix and on the size, the arrangement and the orientation of the elements. In particular, the components of the tensors B and C depend on the internal length parameter, typical of the microstructure ℓ, while the components of the tensor D depend on the square of this length parameter. The tensor A shows no length parameter. [3] In addition, due to the symmetry of $\mathbf{K}_C$ and $\mathbf{K}^r_C$, the continuum material is hyperelastic and $\mathsf{A} = \mathsf{A}^{\mathrm{T}}$, $\mathsf{B}^{\mathrm{t}} = \mathsf{C}$ and $\mathsf{D} = \mathsf{D}^{\mathrm{T}}$, where "T" and "t" are the major and the minor transposition index respectively.

It can be noted that the most common masonry materials, with periodical internal structure, belong at least to the class of centre-symmetrical materials. [4] For these materials the tensors B and C are zero. Considering for example two dimensional assemblies of rectangular blocks of length ℓ and height proportional to this length, the dependence of the components of the tensors A and D from the length

[3] Explicit expressions of the components of the elastic tensors are reported in Trovalusci and Masiani (1999, Appendix A).

[4] These materials have the cyclic group of order two as group of material symmetry.

Table 1
Components of the Cosserat elastic tensors A and D in terms of the parameters ℓ, k_n, k_t, k_{tn}

Parameter	Symbol	
Normal module	A_{1111}	$c_1k_t + c_2k_n$
Normal module	A_{2222}	$c_3k_n + c_4k_t$
Tangential module	A_{1212}	$c_3k_t + c_4k_n$
Tangential module	A_{2121}	$c_1k_n + c_2k_t$
Dilatancy module	A_{1121}	$(c_1 + c_2)k_{tn}$
Dilatancy module	A_{2212}	$(c_3 + c_4)k_{tn}$
Drilling module	D_{121121}	$(c_5k_n + c_6k_t + c_7k_{tn})\ell^2$
Drilling module	D_{122122}	$(c_8k_n + c_9k_t + c_{10}k_{tn})\ell^2$
Drilling module	D_{121122}	$(c_{11}k_n + c_{12}k_t + c_{13}k_{tn})\ell^2$

parameter ℓ is shown in Table 1. [5] In the table k_n, k_t and k_{nt} are respectively the normal, shear and dilatancy stiffness of the joint [6] per unit thickness of the wall, while c_i are constants depending on the arrangement of the bricks.

It is possible to obtain the constitutive relations for the equivalent classical continuum by assigning constitutive prescriptions to the micropolar model. In particular, by requiring that the microrotation velocity, $\mathbf{W}$, corresponds to the macrorotation velocity, skw$\,\mathbf{H}$, [7] and by requiring that the couple-stress, $\mathbf{C}$, be null, a Cauchy model equivalent to the original block system is obtained. [8] The Cauchy symmetric stress tensor also depends on the contact forces and on the geometry of the module [9]

$$\hat{\mathbf{S}}(X) = \frac{1}{V}\sum_{C} \mathrm{sym}(\mathbf{t}_C \otimes (B - A)). \tag{9}$$

By using linear elastic response functions for the forces $\mathbf{t}_C$ (Eq. (7a)) the constitutive relations are obtained as follows

$$\hat{\mathbf{S}} = \hat{\mathsf{A}}(\mathrm{sym}\mathbf{U}). \tag{10}$$

The constitutive terms of Eq. (10) have no intrnal length parameter. Unlike the Cosserat model, the simple (grade one) Cauchy model does not account for the scale effects. Moreover, through the constitutive equations (8) and (10), it can be shown that, in general, the two materials have different symmetry groups.

In a two-dimensional frame the components of the symmetric elastic tensors, $\hat{\mathsf{A}}$, are

$$\begin{aligned}
\hat{\mathsf{A}}_{1111} &= \mathsf{A}_{1111},\\
\hat{\mathsf{A}}_{1122} &= \mathsf{A}_{1122},\\
\hat{\mathsf{A}}_{2222} &= \mathsf{A}_{2222},\\
\hat{\mathsf{A}}_{1112} &= (\mathsf{A}_{1112} + \mathsf{A}_{1121})/2,\\
\hat{\mathsf{A}}_{2212} &= (\mathsf{A}_{2212} + \mathsf{A}_{2221})/2,\\
\hat{\mathsf{A}}_{1212} &= (\mathsf{A}_{1212} + \mathsf{A}_{2121} + 2\mathsf{A}_{1221})/2.
\end{aligned} \tag{11}$$

[5] The numerical indices of a tensor A of order n indicate the component of the tensor in an orthonormal base $\{\mathbf{e}_i\}$, $i = 1, n$: $\mathsf{A}_{ij\cdots n} = \mathsf{A} \cdot \mathbf{e}_i \otimes \mathbf{e}_j \otimes \cdots \otimes \mathbf{e}_n$.

[6] The drilling stiffness of the joint is proportional to $k_n\ell^2$.

[7] The operator "skw" stands for the skew part of a tensor.

[8] Appendix in Masiani and Trovalusci (1996).

[9] The operator "sym" sorts out the symmetric part of a tensor.

3. Cosserat and Cauchy non-linear materials

The procedure of identification of the stress measures of the equivalent micropolar continuum does not depend on the specific response functions required for the interactions among the bodies in the discrete medium. Therefore, the non-linear continuous material can be obtained by describing the internal actions of the Lagrangian system according to proper non-linear constitutive relations. To show how the differences between the response of the Cosserat and the Cauchy models increase in a non-linear frame, we consider a simple case of non-linear behaviour. For blocky materials it is advisable to take into account the weak resistance to tension and the friction at the interfaces. To this aim the interactions among the elements are bounded directly on the interfaces of the elements of the discrete system. In this way, the problem of determining a yield domain in the continuum stress space, generally obtained considering the actions of micromodels according to various standards (Mühlhaus, 1993), is bypassed. The relations among the stress measures of the continuous and the discontinuous model are in fact not simple to formulate, especially if the continuum is provided with microstructure. To see this, look for instance at the space dimension: in the micropolar continuum the dimension of the stress space is eighteen, while in the discrete model the space of the contact actions of each interface has dimension six.

For materials having at least the central symmetry, which, as already noted, correspond to common masonry materials, the tensors B and C in Eq. (8) are zero; using Eqs. (3) and (7), the expressions for the contact actions on the module can be obtained in terms of stress measures of the continuum as follows

$$\begin{aligned} \mathbf{t}_C &= \mathbf{K}_C\big\{(\mathsf{A}^{-1}\mathbf{S})(B-A) + [(\mathsf{D}^{-1}\mathbf{C})(B-X)](C-B) - [(\mathsf{D}^{-1}\mathbf{C})(A-X)](C-A)\big\}, \\ \mathbf{T}_C &= \mathbf{K}_C^r(\mathsf{D}^{-1}\mathbf{C})(B-A). \end{aligned} \tag{12}$$

In a two-dimensional frame, the restrictions imposed on the contact forces of each joint of the module are (Fig. 2)

$$\begin{aligned} &\mathbf{t}_C\cdot\mathbf{n}_C \leqslant a, \\ &|\mathbf{t}_C\cdot\mathbf{m}_C| \leqslant \tan\phi(a-\mathbf{t}_C\cdot\mathbf{n}_C), \\ &|\mathbf{T}_C\cdot\mathbf{n}_C\otimes\mathbf{m}_C| \leqslant d_C(a-\mathbf{t}_C\cdot\mathbf{n}_C), \end{aligned} \tag{13}$$

where $\mathbf{n}_C$ and $\mathbf{m}_C$ are the unit vectors outward normal and tangent to the Cth contact surface, ϕ is the friction angle, d_C the half length of the contact surface, $a\tan\phi$ the intensity of the cohesion force, while d_Ca is the intensity of the cohesion couple.

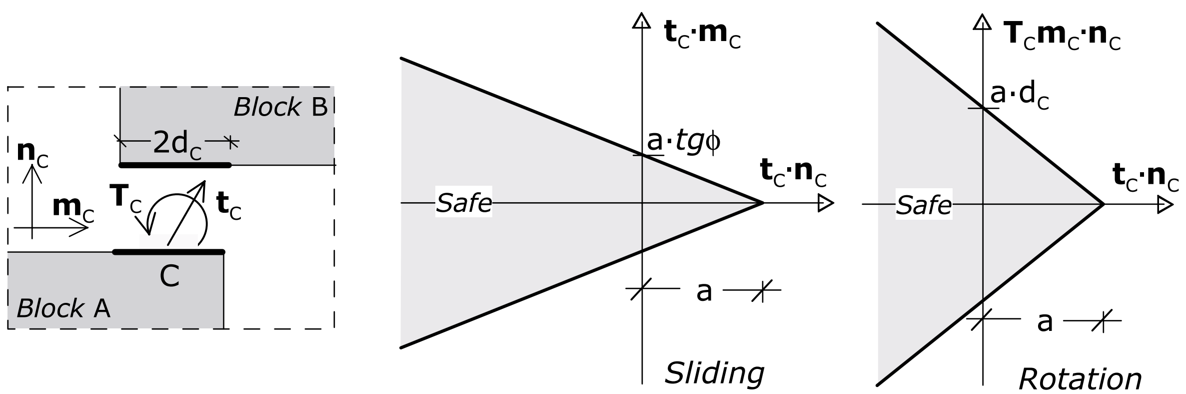

Fig. 2. Yield conditions for the contact actions between two elements of the module.

The first two inequalities (13) are classical. The third inequality represents the condition that, in absence of cohesion, the Cth interface can carry couples until the normal component of force, $\mathbf{t}_C \cdot \mathbf{n}_C$, is applied within the contact surface. Note the presence of the size parameter d_C in the failure criterion. The geometry and the arrangement of the elements are taken into account through the position of the contact points C and the orientation, $\mathbf{n}_C$, of the contact surfaces.

In the standard equivalent continuum, there are only contact forces and the restrictions (13) apply only on the first two.

4. Cosserat and Cauchy solutions

The micropolar continuum provides solutions equivalent to the classical continuum when the work expended by the stress measures of the two models is the same.

If we decompose the strain tensor into $\mathbf{U} = \operatorname{sym}\mathbf{H} + \operatorname{skw}\mathbf{H} - \mathbf{W}$ and use Eq. (8), written for centre-symmetric materials, the work in Eq. (5) becomes

$$p(\mathbf{H} - \mathbf{W}, \mathbf{R}) = \operatorname{sym}(\mathsf{A}(\mathbf{H} - \mathbf{W})) \cdot \operatorname{sym}\mathbf{H} + \operatorname{skw}(\mathsf{A}(\mathbf{H} - \mathbf{W})) \cdot (\operatorname{skw}\mathbf{H} - \mathbf{W}) + \frac{1}{2}\mathsf{D}\mathbf{R} \cdot \mathbf{R}, \tag{14}$$

while in the classical continuum the work writes

$$\hat{p}(\operatorname{sym}\mathbf{H}) = \hat{\mathsf{A}}\operatorname{sym}\mathbf{H} \cdot \operatorname{sym}\mathbf{H}. \tag{15}$$

The tensors $\operatorname{skw}(\mathsf{A}(\mathbf{H} - \mathbf{W}))$ and $\mathsf{D}\mathbf{R}$ represent the non-standard stress components, but, in general, also the symmetric part of the Cosserat stress tensor, $\operatorname{sym}(\mathsf{A}(\mathbf{H} - \mathbf{W}))$, does not coincide with the Cauchy stress tensor, $\hat{\mathsf{A}}\operatorname{sym}\mathbf{H}$, because it depends on $\operatorname{skw}\mathbf{H} - \mathbf{W}$.

When there are no kinematical requirements, the micropolar and the classical solution can be comparable only under specific constitutive prescriptions, ensuring the correspondence of the work. The work of the Cosserat continuum is the same as the work of the Cauchy continuum only when

$$\left.\begin{array}{l} \operatorname{skw}(\mathsf{A}(\mathbf{H} - \mathbf{W})) = \mathbf{0} \\ \operatorname{sym}(\mathsf{A}(\mathbf{H} - \mathbf{W})) = \hat{\mathsf{A}}\operatorname{sym}\mathbf{H} \end{array}\right\} \quad \forall \mathbf{U}, \mathbf{R}. \tag{16}$$

Because of the moment balance equations written in absence of body couples,

$$\operatorname{div}\mathsf{D}\mathbf{R} - 2\operatorname{skw}(\mathsf{A}(\mathbf{H} - \mathbf{W})) = \mathbf{0}, \tag{17}$$

Eq. (16a) is equivalent to

$$\mathsf{D} = \mathbf{0}. \tag{18}$$

Considering for simplicity a two-dimensional frame, the only independent component of the skew part of the micropolar stress tensor, accounting for the major symmetry of A, is

$$\begin{aligned} 2\operatorname{skw}(\mathsf{A}(\mathbf{H} - \mathbf{W}))_{12} = {} & (\mathsf{A}_{1211} - \mathsf{A}_{2111})(\operatorname{sym}\mathbf{H})_{11} + (\mathsf{A}_{1222} - \mathsf{A}_{2122})(\operatorname{sym}\mathbf{H})_{22} + (\mathsf{A}_{1212} - \mathsf{A}_{2121})(\operatorname{sym}\mathbf{H})_{12} \\ & - (\mathsf{A}_{1212} + \mathsf{A}_{2121} - 2\mathsf{A}_{1221})((\operatorname{skw}\mathbf{H})_{12} - \mathsf{W}_{12}), \end{aligned} \tag{19}$$

while the components of its symmetric part are

$$\begin{aligned} \operatorname{sym}(\mathsf{A}(\mathbf{H}-\mathbf{W}))_{11} &= \mathsf{A}_{1111}(\operatorname{sym}\mathbf{H})_{11} + \mathsf{A}_{1122}(\operatorname{sym}\mathbf{H})_{22} + (\mathsf{A}_{1112}+\mathsf{A}_{1121})(\operatorname{sym}\mathbf{H})_{12} \\ &\quad + (\mathsf{A}_{1112}-\mathsf{A}_{1121})((\operatorname{skw}\mathbf{H})_{12}-\mathbf{W}_{12}), \\ \operatorname{sym}(\mathsf{A}(\mathbf{H}-\mathbf{W}))_{22} &= \mathsf{A}_{2211}(\operatorname{sym}\mathbf{H})_{11} + \mathsf{A}_{2222}(\operatorname{sym}\mathbf{H})_{22} + (\mathsf{A}_{2212}+\mathsf{A}_{2221})(\operatorname{sym}\mathbf{H})_{12} \\ &\quad + (\mathsf{A}_{2212}-\mathsf{A}_{2221})((\operatorname{skw}\mathbf{H})_{12}-\mathbf{W}_{12}), \\ 2\operatorname{sym}(\mathsf{A}(\mathbf{H}-\mathbf{W}))_{12} &= (\mathsf{A}_{1211}+\mathsf{A}_{2111})(\operatorname{sym}\mathbf{H})_{11} + (\mathsf{A}_{1222}+\mathsf{A}_{2122})(\operatorname{sym}\mathbf{H})_{22} \\ &\quad + (\mathsf{A}_{1212}+\mathsf{A}_{2121}+2\mathsf{A}_{1221})(\operatorname{sym}\mathbf{H})_{12} - (\mathsf{A}_{1212}-\mathsf{A}_{2121})((\operatorname{skw}\mathbf{H})_{12}-\mathbf{W}_{12}). \end{aligned} \tag{20}$$

The components of the Cauchy stress tensor are

$$\begin{aligned} (\hat{\mathsf{A}}(\operatorname{sym}\mathbf{H}))_{11} &= \hat{\mathsf{A}}_{1111}(\operatorname{sym}\mathbf{H})_{11} + \hat{\mathsf{A}}_{1122}(\operatorname{sym}\mathbf{H})_{22} + 2\hat{\mathsf{A}}_{1112}(\operatorname{sym}\mathbf{H})_{12}, \\ (\hat{\mathsf{A}}(\operatorname{sym}\mathbf{H}))_{22} &= \hat{\mathsf{A}}_{2211}(\operatorname{sym}\mathbf{H})_{11} + \hat{\mathsf{A}}_{2222}(\operatorname{sym}\mathbf{H})_{22} + 2\hat{\mathsf{A}}_{2212}(\operatorname{sym}\mathbf{H})_{12}, \\ 2(\hat{\mathsf{A}}(\operatorname{sym}\mathbf{H}))_{12} &= \hat{\mathsf{A}}_{1211}(\operatorname{sym}\mathbf{H})_{11} + \hat{\mathsf{A}}_{1222}(\operatorname{sym}\mathbf{H})_{22} + 2\hat{\mathsf{A}}_{1212}(\operatorname{sym}\mathbf{H})_{12}. \end{aligned} \tag{21}$$

Eq. (16a) is satisfied when the internal length, ℓ, of the material vanishes with respect to the structural length, L, being in this case the tensor D null, or when the following constitutive prescriptions are satisfied

$$\mathsf{A}_{1211} - \mathsf{A}_{2111} = 0, \tag{22a}$$

$$\mathsf{A}_{1222} - \mathsf{A}_{2122} = 0, \tag{22b}$$

$$\mathsf{A}_{1212} - \mathsf{A}_{2121} = 0, \tag{22c}$$

$$\mathsf{A}_{1212} - \mathsf{A}_{2121} - 2\mathsf{A}_{1221} = 0. \tag{22d}$$

The conditions that yield a coincidence between the Cosserat and the Cauchy solution differ for materials belonging to different class of symmetry.

If the material is at least orthotetragonal [10] Eqs. (22a)–(22c) are verified; whereas neither orthotetragonal materials nor materials belonging to a narrower class of symmetries, like isotropic materials, [11] can satisfy Eq. (22d). Therefore, when the elements have vanishing dimensions the couple-stress is null and the balance of moment (17) requires the vanishing of the skew part of the stress tensor (Eq. (16a)). This implies the condition

$$\operatorname{skw}\mathbf{H} - \mathbf{W} = \mathbf{0}. \tag{23}$$

In this case, if the Cauchy elastic moduli are identified according to Eq. (11), Eq. (16b) is also satisfied and the two solutions coincide. This confirms the well known result that, for isotropic materials, the classical and the micropolar solutions coincide when the internal length vanishes (Mindlin, 1963).

If the material is orthotropic, [12] or belongs to a wider class of symmetry, the vanishing of the internal length is not enough to ensure the correspondence between the two models. This is because the balance equation (17) does not imply the condition (23) and in general Eq. (16b) is not satisfied. In fact, for such anisotropic materials not all the conditions (22a)–(22d) hold and in particular, since they always have two different shear moduli, A_{1212} and A_{2121}, Eq. (22c) cannot be satisfied. [13]

[10] The orthotetragonal symmetry is the symmetry of the square; it corresponds to a texture of juxtaposed square elements with no interlocking. The material symmetry group is the dihedral group of order four. The conditions on the elastic components are reported in Table 1 of the paper Trovalusci and Masiani (1999). Obviously, these conditions are verified for isotropic materials.

[11] For an explicit representation formula of isotropic micropolar materials see Eringen (1968).

[12] The orthotropic symmetry is the symmetry of the rectangle, it is the symmetry of the common masonry made of bricks regularly spaced. The material symmetry group is the dihedral group of order two.

[13] In the common case of orthotropy only Eq. (22a) and (22b) are verified.

When the size of the elements is negligible, the micropolar solution is then comparable to the classical solutions only for materials having at least the orthotetragonal symmetry. In the other cases of material symmetry, the two solutions coincide only if the internal constraint (23) is posed.

When the size of the elements is not negligible, Eq. (16) are verified if the condition (23) is posed and if the microrotation gradient, **R**, is null. This happens for example in case of uniform compression, where the local rigid rotation, skw**H**, is constant and equal to the microrotation, **W**. In this case, the microrotation gradient vanishes, the stress tensor is equal to the standard symmetric tensor and the two solutions correspond, regardless of the internal length of the microstructure.

Except for particular cases of deformations, every time the material is not at least orthotetragonal and the actual value of the length parameter cannot be considered small, as in the case of ordinary masonry, the Cauchy and the Cosserat solutions differ.

Differences between the Cosserat and the Cauchy solutions can be observed when the microrotation gradient, **R**, or the skew part of the strain tensor, skw**H** − **W**, are important. Whenever the microrotation gradient has a significant role and the components of **D** are large, the work of the couple-stress becomes remarkable. In this case, even in the elastic field, the response of the material develops a strong dependence on the ratio between the length parameter of the microstructure, ℓ, and of the macrostructure, L (scale ratio). As the Cauchy model cannot account for the scale effects, the differences between the Cosserat and the Cauchy solutions increase when the scale ratio increases. Otherwise, when the skew part of the strain tensor is important, the work of both the symmetric and the skew part of the stress tensor can be significant. In particular, the work of the micropolar continuum can be as much different from the work of the classical continuum as the sum of the two tangential moduli, A_{1212} and A_{2121}, differ from $2\mathsf{A}_{1221}$.

Further discrepancies between the Cosserat and the Cauchy solution arise because the Cauchy model, in general, does not preserve the material symmetries of the Lagrangian system. Though the components of the classical elastic tensor depend on the shape and arrangement of the elements, the response of the Cauchy material to the variation in the masonry texture and therefore to the variation of the material symmetry class is, compared to Cosserat continuum, less sensitive.

5. The algorithm of solution

The constitutive model obtained by establishing the equivalence of the mechanical power between the discontinuous and the continuous systems using the restrictions (13) is non-linear. The numerical solution of the non-linear problem is obtained by solving a linear incremental problem at each step, using a standard finite elements approach. A finite number of load steps is considered. For each load step, the problem is linearised and the integral equivalence procedure adopted in the linear context is used. Moreover, after the occurrence of yield situations for the contact actions in the discrete system the stiffness matrix is corrected. This procedure can be included among the methods started by Castigliano (1879), based on the progressive reduction of the stiffness in media with limited resistance to specific stresses; in this case the correction of the elastic moduli is deduced from the study of the failure mechanisms in the discrete system rather than from experimental criteria or other kind of research widely used in the past (see for instance Essawy et al. (1985)).

The finite element used to solve the two-dimensional micropolar problem is a plane stress triangular element with three nodes having three degrees of freedom each (two displacement components and one in-plane rotational component). For the degree of rotational freedom a plane shape function was adopted (Masiani and Trovalusci, 1995). For the Cauchy model the finite element used is the standard plane stress element where the stiffness matrix is constructed using the elastic tensor in Eq. (10).

We perform an algorithm working through small load steps as follows.

(i) Given a load increment, the discretised field problem is solved yielding the increase of displacements.

(ii) In the Gauss points of each element, the contact forces of the discrete model are evaluated through the expressions (12).

(iii) The admissibility of the contact actions is verified using the inequalities (13). If these conditions are violated, the constitutive tensors $\mathbf{K}_C$ and $\mathbf{K}^r_C$ are modified by removing the relevant components and the corresponding increases in the contact actions, $\Delta\mathbf{t}_C$ and $\Delta\mathbf{T}_C$, are put to zero.

In a two-dimensional frame the components $k_n = \mathbf{K}_C \cdot \mathbf{n}_C \otimes \mathbf{n}_C$, $k_t = \mathbf{K}_C \cdot \mathbf{m}_C \otimes \mathbf{m}_C$ and $k_{nt} = \mathbf{K}_C \cdot \mathbf{n}_C \otimes \mathbf{m}_C$ of the tensor $\mathbf{K}_C$ are respectively the normal, the shear and the dilatancy stiffness, while the sole independent component of the tensor $\mathbf{K}^r_C$, K^r, is the drilling stiffness.

If the first inequality (13) is violated (opening) then

$$\mathbf{K}_C = \mathbf{0}, \quad \mathbf{K}^r_C = \mathbf{0} \Rightarrow \Delta\mathbf{t}_C = \mathbf{0}, \quad \Delta\mathbf{T}_C = \mathbf{0}, \tag{24}$$

if the second of the (13) is not verified (sliding) then

$$k_t = 0 \Rightarrow \Delta\mathbf{t}_C \cdot \mathbf{m}_C = 0, \tag{25}$$

if the third is violated (rotation) then

$$\mathbf{K}^r_C = \mathbf{0} \Rightarrow \Delta\mathbf{T}_C = 0. \tag{26}$$

(iv) Using again the power equivalence procedure for each finite element, the new elastic tensor of the continuum material equivalent to the "damaged" discrete system is estimated and, with the usual techniques, the new stiffness matrix of the element is evaluated.

The procedure stops at the end of the load path or when it is no longer possible to find a balanced solution of the problem.

6. Test problems

To underline the features of the micropolar model, with particular attention to its differences from the classical continuum, some test problems were solved.

A square two-dimensional masonry panel (size 4×4 m), subjected to several boundary conditions, was studied varying the size of the blocks. The heterogeneous material considered is an orthotropic masonry with regularly alternated bricks and joints of mortar. The mechanical features of the heterogeneous discontinuous material and the elastic constants of the equivalent homogeneous continuum are shown in Table 2. The drilling stiffness moduli of the Cosserat continuum, indicated in Table 3, change with the length of the bricks.

When geometrical or load discontinuities are involved, the micropolar solution is marked by a non-homogeneous microrotation field. Therefore, couple-stresses can play a significant role; considerable scale effects are also observed. As previously noted, only the constitutive relation between the couple-stress and the microrotation gradient explicitly contains the internal length parameter of the microstructure. This means that, when considerable gradients of microrotation are involved, the Cosserat model is affected by scale effects, unlike the Cauchy model. These effects increase considerably when the microstructure size becomes comparable to the macrostructure size. Figs. 3 and 4 show, for example, the results yielded by the micropolar and by the classical model for the linear elastic problem of the panel hinged on a part of the bottom side and subjected to a load distributed on a part of the topside. Assuming the load print as macrostructure length, $L = 1$ m, and said ℓ the internal length, here the length of the brick, three cases were analysed, respectively correspondent to: $\ell/L = 2 \times 10^{-2}$, $\ell/L = 2 \times 10^{-1}$, $\ell/L = 2$. The comparison is made between the vertical components of the displacement of the two models, u_2 (Fig. 3), and between the components of the skew-symmetric Cosserat microrotation tensor, W_{12} (Fig. 4), and of the Cauchy

Table 2
Test problems: mechanical parameters of Lagrangian system and elastic constants of equivalent continua

Parameter	Symbol	Numerical value
Friction angle	$\tan\phi$	0.600
Cohesion	a/d_C	1.000 MN/m
Normal joint stiffness	k_n/d_C	1.500×10^4 MN/m^2
Shear joint stiffness	k_t/d_C	0.750×10^4 MN/m^2
Dilatancy joint stiffness	k_{nt}/d_C	0.000 MN/m^2
Drilling joint stiffness	K^r/d_C	0.375×10^6 MN
Normal module	$\mathbf{A}_{1111} = \widehat{\mathbf{A}}_{1111}$	3.750×10^4 MN/m^2
Normal module	$\mathbf{A}_{2222} = \widehat{\mathbf{A}}_{2222}$	1.500×10^4 MN/m^2
Tangential module	$\mathbf{A}_{1212}$	0.750×10^4 MN/m^2
Tangential module	$\mathbf{A}_{2121}$	3.000×10^4 MN/m^2
Tangential module	$\widehat{\mathbf{A}}_{1212}$	1.875×10^4 MN/m^2

Table 3
Test problems: elastic constants of equivalent continua

		Blocks size (m)		
		0.02×0.01	0.20×0.10	2.00×1.00
Drilling module	$\mathbf{D}_{121121}$	1.125×10^6 MN	112.5×10^6 MN	11250×10^6 MN
Drilling module	$\mathbf{D}_{122122}$	0.375×10^6 MN	37.50×10^6 MN	3750×10^6 MN
Drilling module	$\mathbf{D}_{121122}$	0.000 MN	0.000 MN	0.000 MN

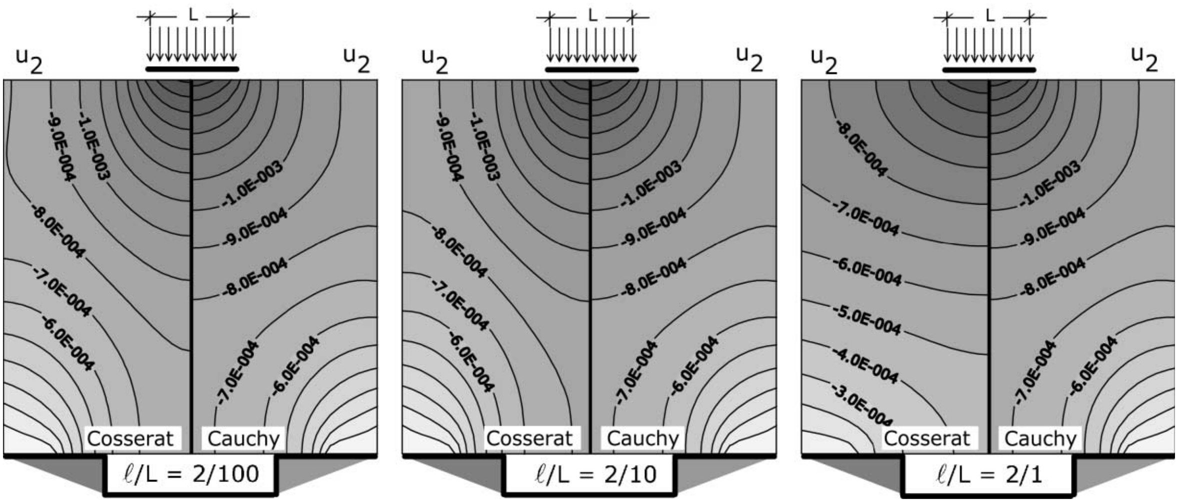

Fig. 3. Vertical components of displacement: contour lines for various scale ratio (ℓ is the brick length).

macrorotation tensor, skw$\mathbf{H}_{12}$. The comparison shows that the discrepancies between the two models tend to increase with the increase of the scale ratio ℓ/L.

From these results one could expect that the influence of the micropolar structure, as it happens for isotropic elastic materials (Mindlin, 1963), becomes negligible for small values of the scale ratio. Another test sample, consisting in a panel resting on its base under a force applied at the middle of the topside, was

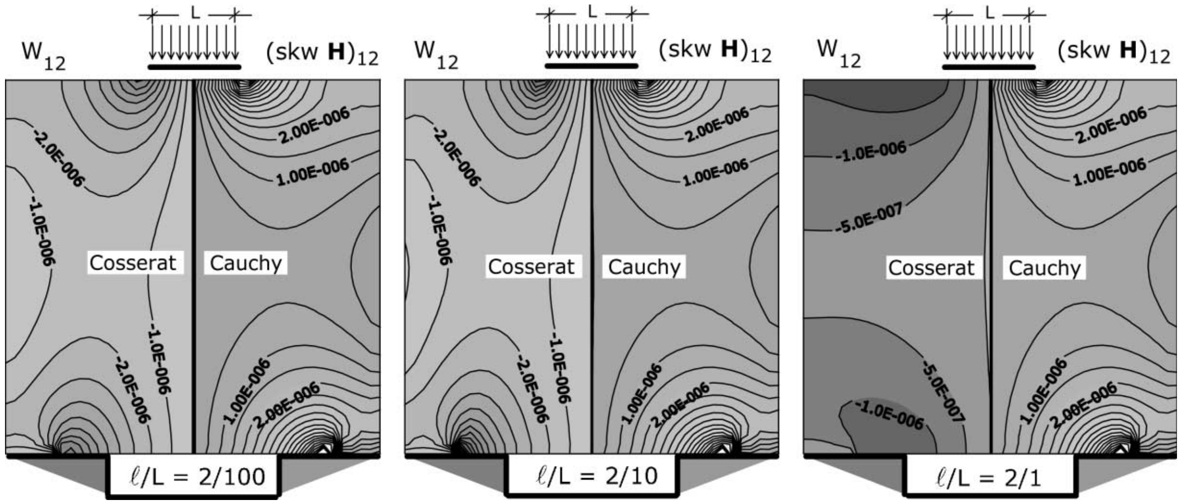

Fig. 4. Components of microrotation (Cosserat) and macrorotation (Cauchy): contour lines for various scale ratio (ℓ is the brick length).

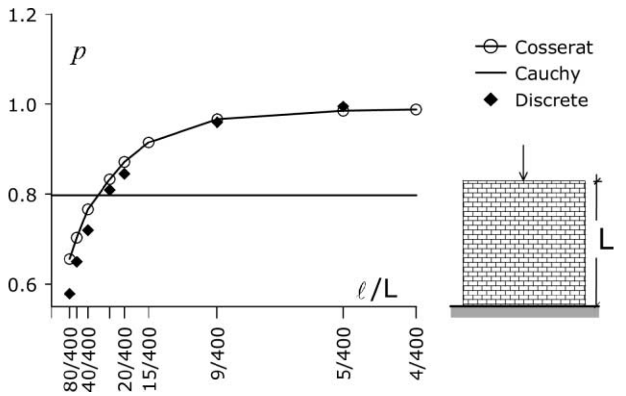

Fig. 5. Normalised strain energy of the Cosserat, Cauchy and discrete solutions for different values of the scale ratio ℓ/L.

performed to show that differences between the two solutions can be appreciated also when the scale ratio decreases. In this case the macrostructure length L is the constant length of the panel and the microstructure length ℓ is the variable length of the bricks. The variation of the (normalised) strain energy of the two models with the scale ratio is shown in the diagram of Fig. 5. For comparison, the results of the Lagrangian model are also reported. As observed in Section 4, being the material orthotropic, the Cosserat solution does not tend to the Cauchy solution when the internal length ℓ goes to zero.

The differences between the classical and the micropolar model increase in non-linear field. This also happens in lack of concentrations of stresses and when the linear solution does not evidence notable microrotation gradients.

Figs. 6 and 7 outline the problem of the wall, hinged on the bottom side, subjected to an inclined body force in the 1:6 ratio between the horizontal (5×10^{-4} MN/m^2) and the vertical component (3×10^{-3} MN/m^2). This is the classical problem of the simplified modelling of seismic actions as static forces. In the linear elastic field, the differences observed between the Cosserat and Cauchy models are slight; in addition, the

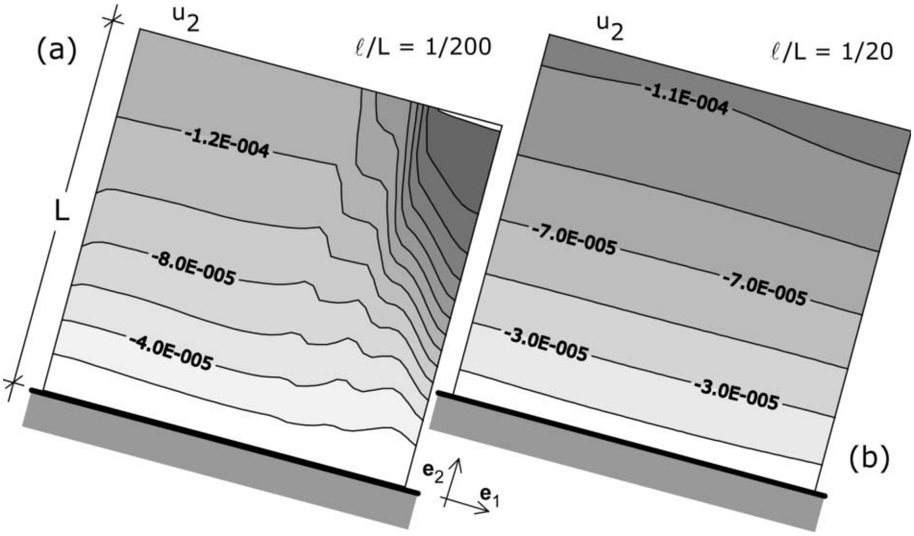

Fig. 6. Cosserat non-linear model. Components u_2 of displacement: contour lines for two values of the scale ratio (ℓ is the brick length). Small bricks (a) and large bricks (b).

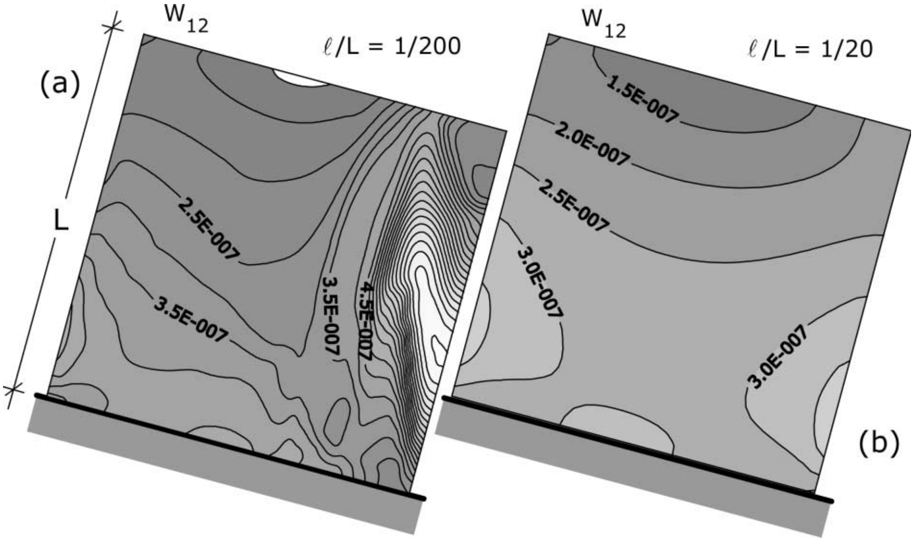

Fig. 7. Cosserat non-linear model. Components of microrotation: contour lines for two values of the scale ratio (ℓ is the brick length). Small bricks (a) and large bricks (b).

size of the blocks is not crucial. These effects are due to the substantial regularity of the solution and to the small values of couple-stresses. The same problem was solved in a non-linear field, using the values reported in Table 2 for the cohesion and the friction angle. The solution is obtained by increasing step by step the horizontal component of force for two different values of the scale ratio ℓ/L: 5×10^{-3} (a), and 5×10^{-2} (b), where the microlength, ℓ, is the brick length and the macrolength, L, is the length of the panel. Fig. 6 shows the contour lines of the vertical components of the displacement fields, u_2, and Fig. 7 shows the contour lines of the components of the microrotation field, W_{12}, with respect to case (a) and (b). In the case of small

Fig. 8. Experimental test: effect of the weight on an inclined brick wall.

bricks (a) a vertical band, on the right of the panel, stands out where there is a concentration of the strains and, in a sense, of the "crack". In the case (b), instead, the larger size of the blocks prevents their relative rotation and their crumbling. The mode of failure of a physical sample, similar to the numerical sample analysed, is shown in Fig. 8. [14] A gross estimate of the "damage" can be made by checking the state of each finite element in the space of the contact actions in the Lagrangian model through Eqs. (12) and (13). The extent of the damage can be estimated by considering the percentage of "cracked" elements. In both cases, the damage percentages obtained are, respectively, of 96% and 38%, while the classical continuum model provides results that do not depend from the size of the blocks. Unlike the linear solution, the scale ratio greatly affects the results, reducing the effectiveness of the classical continuum model. This different behaviour is proved by experimental tests in Baggio and Trovalusci (1993).

7. Final remarks

The mechanical behaviour of discontinuous materials with internal structure can be grossly represented using a continuum model. This continuum must retain memory of the geometry of the actual internal structure and of the size of the elements. The effectiveness of the gross description depends on the kind of continuum selected and in particular on the presence of kinematical terms of order different from one in the strain energy formula. These terms enable to include an internal length parameter in the field equations and to deal with problems in which scale effects are important.

In this work, masonry-like materials are described as micropolar continua and, in order to make comparisons, as classical continua. Linear elastic and non-linear constitutive equations for the two continua are derived from the response functions of a Lagrangian system of interacting rigid bodies using an integral procedure of equivalence in terms of virtual work. The Cosserat continuum proves effective in representing the behaviour of such Lagrangian systems because it allows: to take into account, besides the mechanical

[14] The picture shows the effect of the weight on an inclined wall made of bricks arranged according to an orthotropic texture.

properties of the components, the geometry and the disposition of the units; to discern the behaviour of systems made of elements with different size; to describe the unsymmetries along different planes.

The two continuous models yield correspondent solutions when they expend the same amount of work. Therefore, independently from the specific procedure of equivalence used, remarkable differences of the micropolar solution with respect to the classical solution can be appreciated when the work of the non-standard components of stress, the Cosserat stress tensor—both the skew-symmetric part and a portion of the symmetric part—and the couple-stress tensor, are significant. The two solutions can be comparable in the following cases: when the relative rotations among the elements are not involved, for instance in absence of load or geometrical discontinuities; or when the size of the elements is small with respect to the size of the problem. In the former case, the microrotation gradient, and then the couple-stress, is negligible; in the latter the couple-stress is negligible. In both cases, from balance considerations, also the skew part of the stress tensor vanishes. However, only if the symmetric part of the micropolar stress tensor coincides with the classical stress tensor the work expended by the two models corresponds, and the Cauchy equivalent model can be used to represent satisfactory the discrete assembly. This occurs when the size of the elements is small and the material is at least orthotetragonal, or when the microrotation is equal to the local rigid rotation. Apart from particular cases of deformation or specific material symmetries, the Cosserat and the Cauchy solutions differ and the Cosserat model should be preferred also when the actual values of the scale ratio can be considered small.

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